

Fluctuation-driven directed transport in the presence of Lévy flights

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Abstract

Numerical evidence of directed transport driven by symmetric Lévy noise in time-independent ratchet potentials in the absence of an external tilting force is presented. The results are based on the numerical solution of the fractional Fokker-Planck equation in a periodic potential and the corresponding Langevin equation with Lévy noise. The Lévy noise drives the system out of thermodynamic equilibrium and an up-hill net current is generated. For small values of the noise intensity there is an optimal value of the Lévy noise index yielding the maximum current. The direction and magnitude of the current can be manipulated by changing the Lévy noise asymmetry and the potential asymmetry.

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Fluctuation-driven transport is a topic of considerable theoretical and practical importance. Its simplest realization is the Brownian motion in which fluctuations give rise to diffusion. In the presence of external forces, fluctuations can have a highly nontrivial effect. When inertia is neglected, this problem can be described with an stochastic, overdamped Langevin equations of the form

$$\dot{x} = -\partial_x V + F + \xi(t), \quad (1)$$

where V is the potential, F is an external tilting force, and ξ represents the fluctuations. Of particular interest is the case of periodic potentials, $V(x + L) = V(x)$, lacking reflection symmetry. These potentials, for which there is not an x_0 such that $V(-x) = V(x + x_0)$, are known as ratchets (see Fig. 1). The spatial symmetry breaking implies a preferred direction along which the restoring force on a particle at a potential minima is weaker. Because of this, it is tempting to think that in the presence of fluctuations, particles will tend to drift towards the weak part of the potential giving rise to a “down-hill” unidirectional net transport. However, it can be shown that, consistent with the second law of thermodynamics, such directed transport can not take place in the case of fluctuations in thermodynamic equilibrium (i.e., for ξ corresponding to thermal Gaussian noise), $F = 0$, and time independent potentials, $V = V(x)$. An appealing discussion that illustrates this issue in a ratchet and pawl mechanical device can be found in Ref.[1]. Nevertheless, in the presence of *non-equilibrium* perturbations, fluctuations can give rise to net directed transport in ratchet potentials. The realization and exploitation of this idea has generated a great amount of interest in the non-equilibrium statistical mechanics of “Brownian motors” spanning more than a decade, see for example Ref. [2] and references therein. A trivial way to generate directed transport is by imposing biased perturbations, e.g. a constant F . However, what is highly nontrivial is the appearance of directed currents in the presence of *symmetric*, unbiased, i.e. zero-mean, non-equilibrium perturbations. Referring to Eq. (1), non-equilibrium perturbations can be introduced in F , V and ξ , and ratchet models can be classified depending on how this is done. For example, in “pulsating ratchets” $V = V(x, f(t))$, in “tilting ratchets” a time dependent zero-mean tilting force $F = F(t)$ is assumed, and in “temperature ratchets” the noise strength T in $\langle \xi(t)\xi(s) \rangle = 2\eta k_B T(t)\delta(t - s)$ is a function of time. A less studied class of ratchets, which is the main focus of the present work, incorporate non-equilibrium fluctuations by assuming non-Gaussian noises. Some examples include ratchet transport

driven by symmetric Poissonian white shot noise with exponential distributed amplitudes [3], and the use of colored non-Gaussian noise [4].

In this letter we consider directed transport in ratchet potentials driven by Lévy noise which gives rise to anomalously large particle displacements known as Lévy flights. A Lévy flight process is a non-Gaussian, non-stationary random process whose increments are independent and distributed according to an α -stable Lévy probability distribution function [5]. α -stable distributions play a very important role because, according to the generalized central limit theorem [6], they are the attractors of distributions of sums of random variables with divergent second moments. Also, these distributions exhibit slowly decaying tails describing random processes with large events, and their self-similarity properties make them useful in the description of scale free transport and random fractal processes. These properties explain the ubiquitousness of Lévy statistic in many areas of science, engineering, and economics [7]. The role of Lévy flights in confining potentials has been addressed in the literature before. Examples include the study of the decay properties of the probability distribution function, the study of unimodal-multimodal bifurcations during relaxation [8], and the barrier crossing Kramers problem [9]. However, despite the widespread recognition of the importance of Lévy processes, the effect of Lévy noise in ratchet potentials has not been addressed.

Beyond its intrinsic theoretical interest, one of our motivations for the study presented here is the problem of non-diffusive transport in plasmas. In a recent letter [10] a ratchet-type transport model was proposed to describe non-diffusive impurity transport in magnetically confined fusion plasmas. In that work it was assumed that the fluctuating electrostatic potential is a stationary, homogeneous Gaussian process, and thus in order to introduce a non-equilibrium fluctuation an additional time variation of the two-point Eulerian correlation function had to be assumed. Although these assumptions are reasonable and physical realizable, recent experiments and numerical simulations have shown that there are cases in which fluctuations in turbulent plasmas exhibit Lévy statistics. For example, Lévy statistics has been observed in electrostatic edge turbulence in tokamaks and stellarators [11], and in numerical simulations of pressure-gradient driven plasma turbulence [12]. Accordingly, an open problem of interest is to explore the role of Lévy noise in ratchet-type transport of impurities. Our results provide indirect evidence that in the presence of non-Gaussian Lévy fluctuations, a pinch effect might be present in the case of static potentials. The ideas

presented here might also have an impact on ratchet transport in biological systems and condense matter in general where there is evidence of non-Gaussian fluctuation phenomena.

Our methodology consists of two complimentary approaches based on the Langevin equation (1) driven by Lévy noise, and in the solution of the corresponding space-Fractional Fokker Planck (FFP) equation. For a discussion on the equivalence of the two formulations see Ref. [13]. Although the numerical methods for the solution of fractional differential equations is a rapidly developing field, the integration of the FFP equation with fractional derivatives in space and periodic potentials has not been treated before. In Ref. [14] a numerical study was presented of the FFP equation in a tilted periodic potential in the presence of subdiffusion. The key difference between that work and our contribution is that while Ref. [14] deals with subdiffusive processes caused by non-Markovian memory effects modeled with fractional derivatives in time, here we consider superdiffusive processes caused by Levy flights modeled with fractional derivatives in space. The ratchet potential is given by

$$V = V_0 \begin{cases} 1 - \cos[\pi x/a_1] , & 0 \leq x < a_1 \\ 1 + \cos[\pi(x - a_1)/a_2] , & a_1 \leq x < L , \end{cases} \quad (2)$$

where V_0 is the amplitude, $L = a_1 + a_2$ is the period, $V(x + L) = V(x)$, and $A = (a_1 - a_2)/L$ is the asymmetry parameter. In all the calculations presented here, $V_0 = L = 1$. Compared with the potential $V = V_0[\sin(2\pi x/L) + 0.25 \sin(4\pi x/L)]$ typically used in the literature, e.g. Ref [2], Eq. (2) offers the advantage of an easier control of the spatial asymmetry. As shown in Fig. 1, for $A = -0.274$ both potentials coincide. In the Langevin approach we assume that the noise satisfies $L(\Delta t) = \int_t^{t+\Delta t} \xi(t') dt'$ where L is a Lévy flight process with characteristic function, $\hat{p}_L(k, \Delta t) = \int_{-\infty}^{\infty} e^{ikx} p_L(x, \Delta t) dx$ given by [5]

$$\hat{p}_L(k, \Delta t) = \exp \{ -\Delta t \chi |k|^\alpha [1 - i \text{sign}(k) \beta \tan(\pi\alpha/2)] \} , \quad (3)$$

where α is the Lévy index, β is the skewness parameter, and χ is the Lévy noise intensity. Unless explicitly indicated, we will consider $\chi = 1/2$. Since for the ratchet problem the mean (first moment) of the displacement must be finite, the range for admissible α is $1 < \alpha \leq 2$, whereas $-1 \leq \beta \leq 1$. The time step used in Langevin simulation was $\Delta t = 10^{-3}$. Details of the numerical scheme can be found in Ref. [15]. For $\alpha = 2$, Eq. (3) reduces to a Gaussian corresponding to a diffusive (Brownian) process. As expected, in this case the problem reduces to the Smoluchowski-Feynman ratchet which, as the insert in Fig. 1 shows, only

exhibits a current in the trivial case of a biased constant force F .

One of the main obstacles in the numerical integration of equations containing fractional derivatives in space is the fact that the left, ${}_a D_x^\alpha$, and the right, ${}_x D_b^\alpha$, Riemann-Liouville fractional derivatives are in general singular at the $x = a$ and the $x = b$ boundaries respectively [16]. Here we circumvent this problem by regularizing the fractional space derivatives using the Caputo prescription, ${}_a^c D_x^\alpha P = {}_a D_x^\alpha [P(x) - P(a) - P'(a)(x - a)]$ and ${}_x^c D_b^\alpha P = {}_x D_b^\alpha [P(x) - P(b) + P'(b)(b - x)]$, and write the FFP equation as

$$\partial_t P = \partial_x [P \partial_x V_{eff}] + \chi [l_a^c D_x^\alpha + r_x^c D_b^\alpha] P, \quad (4)$$

where $V_{eff} = V(x) - Fx$, and

$${}_a^c D_x^\alpha P = \frac{1}{\Gamma(2 - \alpha)} \int_a^x \frac{P''(y)}{(x - y)^{\alpha-1}} dy, \quad (5)$$

for $1 < \alpha < 2$, with $P'' = \partial_y^2 P(y, t)$. The expression for ${}_x^c D_b^\alpha P$ follows from Eq. (5) by changing the integration limits to \int_x^b and interchanging x and y in the denominator. The factors $l = -(1 + \beta)/[2 \cos(\alpha\pi/2)]$ and $r = -(1 - \beta)/[2 \cos(\alpha\pi/2)]$ determine the relative weight of the left and the right fractional derivatives. The order of the fractional operators α , the asymmetry parameter β , and the diffusivity χ are given by the index, the skewness, and the scale factor of the characteristic function of the Langevin Lévy noise in Eq. (3). For the numerical integration of Eq. (4) we used a finite difference scheme based on the Grunwald-Letnikov representation of the regularized fractional operators. Further details of the method can be found in Ref. [17]. The integration domain, $x \in (a, b) = (-10, 10)$, covered twenty periods of the potential. We used a grid size $\Delta x = 0.02$ and an integration time step $\Delta t = 1.8 \times 10^{-3}$. The boundary conditions were $P(a) = P(b) = 0$, and the initial condition was a highly peaked normalized Gaussian distribution centered at $x = 0$.

Figure 2 shows typical trajectories obtained from the Langevin equation for different Lévy noise indices α and degrees of potential asymmetry, A . Panel (a) illustrates the absence of directed current in the Gaussian case, whereas panel (b) shows a positive bias in the presence of symmetric Lévy flights. As expected, the Lévy noise gives rise to large particle jumps encompassing several periods. Conclusive evidence of the existence of a net directed current due to symmetric Lévy flights is provided in Fig. 3. Consistent with Fig. 2(b), a net positive current is observed. Note that, as in the case of Poissonian noise [3], “up-hill” transport, i.e. transport the direction of the larger potential gradient, is observed. The dependence of

the steady state current on the asymmetry parameter A for $F = 0$ is illustrated in Fig. 4. Excellent agreement between the Langevin and the Fokker-Planck results is observed. For a fixed value of the Lévy index α , the current is a monotonically decreasing function of the asymmetry and, as expected, for fixed A the current becomes smaller as α approaches the Gaussian limit $\alpha = 2$. Consistent with the symmetry $\partial_x V(x, A) = -\partial_x V(L - x, -A)$, the curves exhibit odd symmetry with respect the origin. Figure 5 shows the dependence of the steady state current on the Lévy index α in the $\beta = 0$ symmetric case. For large χ , the current is a monotonically decreasing function of α . However, for small χ the relation is non-monotonic and, depending on A , there is an optimal value of α that yields the maximum current. It is also instructive to explore the variation of the current with α fixing the scale factor $\sigma = \chi^{1/\alpha}$ of the Lévy distributions in Eq.(3). In this case the dependence seems to be always monotonic. A problem of interest in ratchet transport is the control of the magnitude and direction of the current. Figure 6 shows how this can be accomplished by manipulating the Lévy noise asymmetry β and the potential asymmetry A . Consistent with the Langevin simulation in Fig. 2(c), in Fig. 6(a) there is a net current with $A = 0$ and $\beta = 0.25$. Comparing with Fig. 3, for which $A = -0.274$ and $\beta = 0$, it is observed that the noise asymmetric can mimic the potential asymmetry. As Fig. 6(b) shows, for a given ratchet potential, it is possible to find a compensating value of β for which the current vanishes. Moreover, as Fig. 6(c) shows, a further increase of the Lévy noise skewness leads to a current reversal, a result consistent with the corresponding Langevin simulation shown in Fig. 2 (d).

Summarizing. In this letter we have presented numerical evidence of directed transport driven by symmetric Lévy noise in time-independent ratchet potentials in the absence of an external tilting force. In the limit $\alpha = 2$, the noise becomes Gaussian, the fluctuations are in thermodynamic equilibrium, and consistent with the second law of thermodynamics the current vanishes. However, for $\alpha \neq 2$, the Lévy noise drives the system out of thermodynamic equilibrium and an up-hill net current is generated. For small values of χ and a fixed potential asymmetry, there is an optimal value of α yielding the maximum current. The direction and magnitude of the current can be manipulated by changing the Lévy noise asymmetry and the potential asymmetry. As an application, we conjecture that a recently proposed ratchet pinch mechanism in magnetically confined plasmas might be present even in the case of static electrostatic fluctuations.

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FIGURE CAPTIONS

FIG. 1. Effective ratchet potential $V_{eff} = V(x) - Fx$ for $F = -1$ according to Eq. (2), with $A = -0.274$ (dashed line), $A = 0.2$ (dotted line), and $A = -0.5$ (dashed-dotted line). The solid line corresponds to the ratchet potential discussed in Ref. [2]. (see Fig.2.4) . The inset shows the well-known dependence on F in the $\alpha = 2$ Gaussian case according to the Langevin simulations (dots) with $A = -0.274$, and the result reported in Ref. [2] (solid line).

FIG. 2. Typical trajectories according to the numerical solution of the overdamped Langevin Eq.(1) with $L = 1$, $\chi = 0.5$ and: (a) $\alpha = 2$ (Gaussian), $A = -0.274$; (b) $\alpha = 1.50$, $\beta = 0$, $A = -0.274$; (c) $\alpha = 1.50$, $\beta = 0.25$, $A = 0$; (d) $\alpha = 1.50$, $\beta = -0.5$, $A = -0.274$.

FIG. 3. Evidence of ratchet current in the Fractional Fokker-Planck model with $\alpha = 1.5$, $\beta = 0$, $\chi = 0.5$, $F = 0$ and $A = -0.274$. The solid line in the main figure shows the mean $\langle x \rangle$ with respect to $x = 0$ which follows a linear scaling (dashed line) corresponding to a current $\langle \dot{x} \rangle = 0.145$. The inset at the top shows the final and initial (dashed line) probability distribution function superimposed with the ratchet potential in Eq. (2). The inset at the bottom shows the profile of the probability distribution function at the final time.

FIG. 4. Steady current as a function of ratchet potential asymmetry A , for $\beta = 0$, $\chi = 0.5$ and Lévy indices $\alpha = 1.5$, $\alpha = 1.75$ and $\alpha = 1.90$. The solid line with dots denote the results according to the Langevin model and the circles and crosses the Fractional Fokker-Planck results. The inset shows the dependence of the current on F for $A = 0.2$ and $A = -0.2$. The curve in the middle is the Gaussian case, which shows no dependence on the sign of A . The top (bottom) curve is the $\alpha = 1.5$ Lévy result for $A = -0.2$ ($A = 0.2$).

FIG. 5. Steady state current $\langle \dot{x} \rangle$ versus Lévy index α for (a) $\chi = 0.5$, (b) $\chi = 0.05$, (c) $\sigma = 0.707$, and (d) $\sigma = 0.0707$. Curves 1, 2, 3, and 4 correspond to $A = -0.274$, $A = -0.4$, $A = -0.5$ and $A = -0.6$ respectively.

FIG. 6. Dependence of ratchet current on Lévy noise asymmetry in the fractional Fokker-

Planck equation. Panels (a), (b) and (c) show the probability distribution function for $\alpha = 1.5$ and different values of β , and A . The fourth panel shows the corresponding time evolution of the mean $\langle x \rangle$.

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